

Geometric phases for two-mode squeezed state

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Abstract

Although the geometric phase for one-mode squeezed state had been studied in detail, the counterpart for two-mode squeezed state is vacant. It is evaluated explicitly in this paper. Furthermore, the total phase factor is in an elegant form, which is just identical to one term of product of two squeezed operators. In addition, when this system undergoes cyclic evolutions, the corresponding geometric phase is obtained, which is just the sum of the counterparts of two isolated one-mode squeezed state. Finally, the relationship between the cyclic geometric phase and entanglement of two-mode squeezed state is established.

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I. INTRODUCTION

Squeezed light plays an important role in the development of quantum optics [24]. It preserves the minimum uncertainty and exhibits non-classical nature of light, such as sub-Poissonian statistics which can be observed as photon antibunching effect. It also has many applications in optical communications and detection of gravitational radiation. It was generalized to nonlinear case by Kwek and Kiang [10]. But their studies were just confined to one mode case. Moreover, two mode squeezed state was studied by Caves and Schumaker systematically [4, 17].

Since geometric phase had been discovered by Berry [2] in the quantum system which underwent adiabatic and unitary evolution, its research exploded. Subsequently, It was extended to non-Abelian case by Wilczek and Zee [25]. Its nonadiabatic and cyclic counterpart [1, 27] was studied by Aharonov and Anandan. Soon, by getting rid of the condition of cyclic evolution, it was generalized to a more general case by Samuel and Bhandari [16], who depended on Pancharatnam's earlier study [15]. Subsequently, using kinematic approach, geometric phase was derived by Samuel and Bhandari [16].

Moreover, the geometric phases also had other more generalization, such as off-diagonal ones [9, 12, 13] and mixed state counterparts [20–22].

In addition, geometric phases also have many applications, which range from quantum information and computation science [7, 8] to condensed matter physics [26]. These contexts are covered by many monographs [3, 6, 19].

Meantime, the interdisciplinary between quantum optics and geometric phase has also been researched. Berry phase for coherent and squeezed states was researched by Chaturvedi, Sriram and Srinivasan [5]. The nonadiabatic geometric phase for squeezed state was studied by Liu, Hu and Li [11]. The geometric phase for nonlinear coherent and squeezed state in kinematic approach was discussed by Yang et. al. [28]. However, the above studies are all confined to one-mode case. As to seek for theoretical progress, the two-mode case will be researched in this paper. Moreover, the degree of entanglement between the two-mode is to be evaluated.

This paper is organized as follows. In the next section, the features of two-modes squeezed states and the kinematic approach to geometric phase will be reviewed. In Sec. III, the geometric phase for two-mode squeezed state is to be calculated. From the above outcome,

when the system undergoes cyclic evolution, the corresponding result is also to be obtained. Moreover, the Von Neumann entropy is going to be calculated. And its relation with geometric phase will also be established. Finally, a conclusion is drawn in the last section.

II. REVIEW OF TWO-MODE SQUEEZED STATES AND GEOMETRIC PHASES

The Hamiltonian for two-mode of electromagnetic field [18] takes the form,

$$H_0 = \Omega(a_+^\dagger a_+ + a_-^\dagger a_-) + \epsilon(a_+^\dagger a_+ - a_-^\dagger a_-), \quad (1)$$

where $\Omega \pm \epsilon$ are the frequencies for the two-mode and we take $\hbar = 1$ for simplicity. Furthermore, Ω and ϵ can be regarded as a carrier frequency and a modulation frequency respectively. And the electromagnetic field are quantized by the following commutation relations

$$\begin{aligned} [a_+, a_-] &= [a_+^\dagger, a_-^\dagger] = 0 \\ [a_+, a_+^\dagger] &= [a_-, a_-^\dagger] = 1. \end{aligned}$$

The squeezed operator [18] is generalized to be

$$S(r, \varphi) \equiv \exp[r(a_+ a_- e^{-2i\varphi} - a_+^\dagger a_-^\dagger e^{2i\varphi})], \quad (2)$$

where the real number r is called the squeeze factor and φ is a real phase angle. Moreover, the above operator (2) is unitary,

$$S^{-1}(r, \varphi) = S^\dagger(r, \varphi) = S(-r, \varphi).$$

Hence, the squeezed vacuum state is

$$S(r, \varphi)|0\rangle. \quad (3)$$

Under the Hamiltonian (1), it evolves as

$$\begin{aligned} e^{-iH_0 t} S(r, \varphi)|0\rangle &= e^{-iH_0 t} S(r, \varphi) e^{iH_0 t} |0\rangle \\ &= S(r, \varphi - \Omega t)|0\rangle \end{aligned} \quad (4)$$

which uses the following formulas [18]

$$\exp[-i\epsilon t(a_+^\dagger a_+ - a_-^\dagger a_-)] S(r, \varphi) \exp[i\epsilon t(a_+^\dagger a_+ - a_-^\dagger a_-)] = S(r, \varphi)$$

and

$$\exp[-i\theta(a_+^\dagger a_+ + a_-^\dagger a_-)]S(r, \varphi) \exp[i\theta(a_+^\dagger a_+ + a_-^\dagger a_-)] = S(r, \varphi - \theta).$$

The geometric phases γ [14] for arbitrary time t takes the form

$$\gamma = \arg\langle\psi(0)|\psi(t)\rangle + \int_0^t \langle\psi(\tau)|H|\psi(\tau)\rangle d\tau. \quad (5)$$

It is physical reality ,due to it is invariant under gauge transformation. And it is can be explained as outcome of parallel transportation in the framework of fiber bundle, i.e., holonomy. That's the reason that it deserves a name called Geometric Phase.

III. EVOLUTIONS OF THE GEOMETRIC PHASE FACTOR

For convenience, instead of calculating the geometric phase, we evaluate the geometric phase factor,

$$e^{i\gamma} = \frac{\langle\psi(0)|\psi(t)\rangle}{\|\langle\psi(0)|\psi(t)\rangle\|} e^{i\delta}, \quad (6)$$

where

$$\delta = \int_0^t \langle\psi(\tau)|H|\psi(\tau)\rangle d\tau \quad (7)$$

which is identical to negative the dynamical phase.

At first, let's calculate the inner product

$$\begin{aligned} \langle\psi(0)|\psi(t)\rangle &= \langle 0|S^\dagger(r, \varphi)e^{-iH_0 t}S(r, \varphi)|0\rangle \\ &= \langle 0|S^\dagger(r, \varphi)S(r, \varphi - \Omega t)|0\rangle, \end{aligned} \quad (8)$$

which uses Eq. (4). In order to work out the total phase, the following formula [18] is very useful

$$S^\dagger(r', \varphi')S(r'', \varphi'') = e^{-i\Theta}S(R, \Phi - \Theta)R(\Theta), \quad (9)$$

where the above parameters satisfy the matrix equation

$$C_{R, \Phi} e^{i\Theta\sigma_3} = C_{r'', \varphi''} C_{-r', \varphi'}, \quad (10)$$

where the matrix $C_{r, \varphi}$ is defined by

$$C_{r, \varphi} = \begin{pmatrix} \cosh r & e^{2i\varphi} \sinh r \\ e^{-2i\varphi} \sinh r & \cosh r \end{pmatrix} \quad (11)$$

and σ_3 is the famous Pauli matrix in the z direction. By substituting Eq. (9) into Eq. (8), one obtains

$$\langle\psi(0)|\psi(t)\rangle = \langle 0|e^{-i\Theta}S(R, \Phi - \Theta)R(\Theta)|0\rangle,$$

where

$$R(\Theta) = \exp[-i\Theta(a_+^\dagger a_+ + a_-^\dagger a_-)].$$

By use of the explicit decomposition of squeezed operator [18]

$$S(R, \Psi) = (\cosh R)^{-1} e^{-a_+^\dagger a_-^\dagger e^{2i\Psi \tanh R}} e^{-(a_+^\dagger a_+ + a_-^\dagger a_-) \ln(\cosh R)} e^{a_+ a_- e^{-2i\varphi \tanh R}}, \quad (12)$$

the total phase can be transformed to be an elegant manner

$$\langle\psi(0)|\psi(t)\rangle = (\cosh R)^{-1} e^{-i\Theta}, \quad (13)$$

of which parameters are determined by Eq. (10). Its explicit form is

$$\begin{pmatrix} e^{i\Theta} \cosh R & e^{i(2\Phi-\Theta) \sinh R} \\ e^{i(\Theta-2\Phi) \sinh R} & e^{-i\Theta} \cosh R \end{pmatrix} = \begin{pmatrix} e^{-i\Omega t(i \cosh 2r \sin \Omega t + \cos \Omega t)} & e^{i(2\varphi-\Omega t-\pi/2) \sinh 2r \sin \Omega t} \\ e^{-i(2\varphi-\Omega t-\pi/2) \sinh 2r \sin \Omega t} & e^{i\Omega t(\cos \Omega t - i \sin \Omega t \cosh 2r)} \end{pmatrix}.$$

Therefore, the element (2, 2) can tell us the total phase factor $e^{-i\Theta}$, which take the form

$$\frac{e^{i\Omega t(\cos \Omega t - i \sin \Omega t \cosh 2r)}}{(\cos^2 \Omega t + \sin^2 \Omega t \cosh^2 2r)^{1/2}}. \quad (14)$$

Moreover, let us calculate another term δ (7) in the expression of geometric phase (5). by substituting Eq. (4) into Eq. (7), one can obtain

$$\delta = \int_0^t \langle 0|S^\dagger(r, \varphi - \Omega\tau)HS(r, \varphi - \Omega\tau)|0\rangle d\tau.$$

By use of the following formulas [18]

$$\begin{aligned} S^\dagger(r, \eta)a_+S(r, \eta) &= a_+ \cosh r - a_-^\dagger e^{2i\eta} \sinh r \\ S^\dagger(r, \eta)a_-S(r, \eta) &= a_- \cosh r - a_+^\dagger e^{2i\eta} \sinh r, \end{aligned}$$

the formula for δ can be simplified as

$$\delta = 2\Omega t \sinh^2 r. \quad (15)$$

Finally, by inserting Eq. (14) and (15) into Eq. (6), the geometric phase is achieved as

$$e^{i\gamma} = \frac{e^{i\Omega t \cosh 2r} (\cos \Omega t - i \sin \Omega t \cosh 2r)}{(\cos^2 \Omega t + \sin^2 \Omega t \cosh^2 2r)^{1/2}}$$

Now, the cyclic geometric phase will be discussed. From the total phase factor (8), it is not hard to see that when $\Omega t = 2\pi$, the state (4) will undergo a genuine cyclic evolution of which the final state is exactly the initial state. In another word, the total phase can be regarded as zero. hence the geometric phase can be explicitly expressed

$$\gamma_c = 4\pi \sinh^2 r \mod 2\pi, \quad (16)$$

which is exactly negative the dynamical phase. Because total phase vanish and the geometric phase is equal to the difference between the total phase and dynamical phase. In addition, from Ref. [28], the geometric phase for isolated one-mode squeezed state is

$$\gamma_{ic} = 2\pi \sinh^2 r \mod 2\pi,$$

where the subscript index i denotes for mode. So if we confine cyclic geometric phase to a simple form, combining with Eq. (16), $\gamma_c = \gamma_{1c} + \gamma_{2c}$, which reveals the addition relationship between the two-mode system and the isolated one mode system.

Moreover, the cyclic geometric phase γ_c is related to the Von Neumann entropy which can measure the entanglement between the two modes in the squeezed state. In order to establish the relationship. Let's calculate the entropy first. By use of Eq. (12),

$$S(r, \varphi - \Omega t)|0\rangle = \frac{1}{\cosh r} \sum_{n=0}^{\infty} (-e^{2i(\varphi - \Omega t)} \tanh r)^n |n\rangle_+ |n\rangle_-.$$

Fortunately, it is already in the form of Schmidt decomposition. By a brute force calculation, the entropy reads

$$E = \cosh^2 r \ln(\cosh^2 r) - \sinh^2 r \ln \sinh^2 r, \quad (17)$$

which is identical to the result in Ref. [23]. Finally, substituting Eq. (16) into Eq. (17), we obtain

$$E = \left(1 + \frac{\gamma_c}{4\pi}\right) \ln\left(1 + \frac{\gamma_c}{4\pi}\right) - \frac{\gamma_c}{4\pi} \ln \frac{\gamma_c}{4\pi},$$

which shows the relationship between the entanglement and the cyclic geometric phase. And the corresponding graph is in Fig. (1).

IV. CONCLUSIONS AND ACKNOWLEDGEMENTS

In this article, the geometric phase factor for two-mode squeezed state is evaluated explicitly. The total phase factor (13) is turned to be an elegant outcome, which is just one term

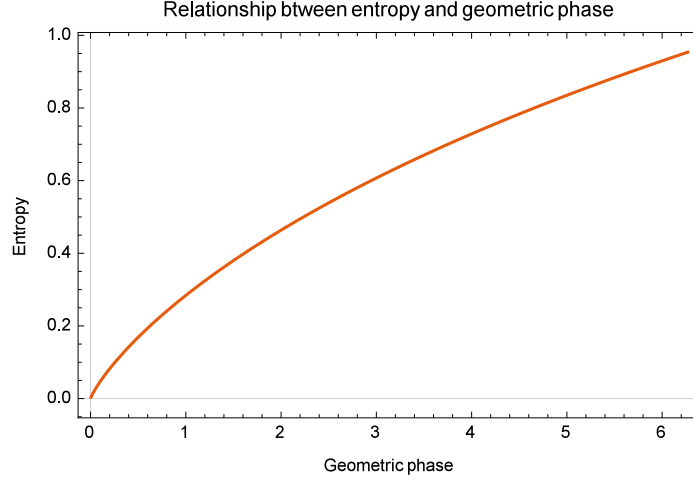


Figure 1: γ_c is set to vary from 0 to 2π .

of the product of the initial squeezed operator and final squeezed operator (9). When this system undergoes cyclic evolutions, the corresponding geometric phase is obtained, which is just the sum of the counterparts of two isolated one-mode squeezed state. Furthermore, the relationship between the cyclic geometric phase and entanglement of two-mode squeezed state is established.

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